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Competent novel strategies for Strong convergence with variational inequality and split equality crisis in fixed point theory

Sajid Anwar[1]*, D.R. Sahu[2*]

[1]Anjuman College of Engineering and Technology, Sadar, Nagpur, Maharashtra 440001, India. [2]Banaras Hindu University, Bazardiha, Maheshpur, Varanasi, Uttar Pradesh 221002, India

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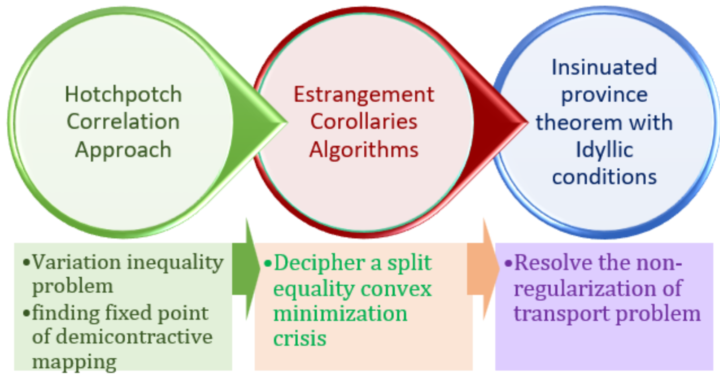
Article

ABSTRACT



The fixed-point theory plays an important role in mathematics, including optimization, mathematical models and economic theories. This study provides new ways to overcome the challenges in the existing fixed-point theories by introducing a new Hotchpotch Correlation Approach for resolving the variation inequality problem and locating the fixed point of demi-contractive mapping. After resolving the variation inequality and the Fixed Points Problems, it incorporates the Estrangement Corollaries Algorithm, which is integrated with the inverse problem and numerical examples that are more important to understand a split equality convex minimization crisis. This increases the consistency and accuracy of the proposed work. To solve the non-regularization of the transport problem, an implied province theorem with necessary and sufficient idyllic conditions is used by deriving the restricted distributions among probability distributions. Moreover, the empirical regularized transport plan system asymptotically follows a Gaussian rule to regularize the entropy and create a limitation law for the extensively allowable Sinkhorn divergence. This suggested approach achieves strong convergence, higher consistency, and precision in the applications of fixed-point theory.

Keywords: fixed-point theory, convergence theorem, optimal transport distances, Gaussian law, empirical regularized transport plan, non-regularization of transport problem

INTRODUCTION

In the 20th century, the theoretical foundation of the fixed-point theory was developed. The fundamental result of this theory is the contraction principle of Picard-Banach-Caccioppoli (from the '30s), which created important lines of research and the theory's applications[1] to functional, differential, integral, etc. The theorems of Tarki, Bourbaki, Banach, Perov, Luxemburg-Jung, Brower, Schauder, Tihonov, and Brouwder-Ghode-Kirk are classic theorems of this theory.[2]

In the theory of metric spaces, Banach's fixed point theory, also known as the contraction principle, is an important tool.[3] For a broad range of applications, it guarantees the presence and uniqueness of solutions to equations of the form x = f(x), f, and provides a constructive method for evaluating these solutions. Stefan Banach (1892-1945), the founder of functional analysis, developed and demonstrated the theorem in 1922.[4] The theory of contraction is an abstract variant of the successive approximation method; in order to solve numerical equations, the method was used empirically from antiquity,[5,6] and it was successfully used for example, to solve Kepler's equation, E = M + e sin E, to determine the position of the planets in orbit (E0=M, E1=M+esin(E0), ..., En=M+e sin(En-1)). Kepler's equations,[7] using the eccentricity e of the orbit and the mean anomaly M, are used to measure the position of objects in our solar system. The eccentric anomaly describes E.

The variational inequality problem (VIP) is defined as:

$$Find\ y \in D \quad such that \quad \langle By, x-y \rangle \geq 0, \qquad \forall x \in D \quad (1)$$

A: C → H is a nonlinear operator denoted by the set of solutions of VIP (1) by V I(D, B). The VIP is an important instrument in economics, decision making,[8] engineering mechanics,

*Corresponding Author: Sajid Anwar
Email: sajidanwar0616@gmail.com

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mathematical programming, transportation, operational analysis, etc. Several iterative methods for solving the VIP and its associated optimization issues[9] have been implemented. The additional gradient method is one of the earliest methods for resolving VIP. The method of the extra gradient was specified as follows:

$$\begin{cases} y_1 \in D, \\ x_j = Q_D\left(y_j - \beta B y_j\right), \\ y_{j+1} = Q_D\left(y_j - \beta B x_j\right), \ j \geq 1, \end{cases} \quad (2)$$

The sub-gradient extra gradient algorithm enhances the extra gradient algorithm to effectively solve the VIP in a real Hilbert environment.[10] Outcomes from prior studies, a weak convergence result for solving the VIP, are obtained under some mild assumptions. More calculation of estimates on the feasible set is needed for the current updated algorithm.[11] Unless the conceivable set has a complex structure that can influence the use of the algorithm, this can be expensive.

In addition to this split common fixed point problem (SCFPP) is defined as

$$find \quad z \in G(U_1) \quad such\,that \quad By \in G(U_2) \quad (3)$$

For the case in which U1 and U2 are non-expansive mappings, the SCFPP studied the setting of Hilbert spaces.[12] They suggested the following algorithm and, under some reasonable conditions, proved its slow convergence to a solution of (3). Thus, it is introduced in the framework of Hilbert spaces for firmly non-expansive operators.[13] To solve this problem, a positive, non-decreasing sequence-based alternative algorithm has been included. Note that those prior algorithms depend on prior knowledge of the operator norms for their implementation.[14] Alongside increasing conceptual appeal and its first practical success in multiple statistical applications,[15,16] there is still no routine use of optimal transfer (OT) based data analysis as it is seriously impeded by its computational burden for many real-world applications.[17] However, their respective average runtime yields a significant limitation for real-world cases.

Nevertheless, as demonstrated in this paper, presents the unified algorithm for the variation inequality problem and the difficulty of finding the fixed point of demicontractive mapping. Along with this, the work includes the generalized algorithm depending on step size so that implementing the novel algorithm does not require prior knowledge of the operator norms. In addition, regularization allows for a thorough descriptive statistics of the corresponding regularization optimal transport plan, a challenge for the (non-regularized) OT plan that is currently out of reach. The ROT plan encodes more structural information across scales than any OT-dependent distance (regularized or not), thus serving as a more informative instrument for inferential statistics. Hence, from above mentioned considerations, there is no literatures concentrates on the fixed point theory based on the variation inequality problem and difficulty of finding the fixed point of demi contractive mapping, split equality convex minimization crisis, non-regularization of transport problem, though there is a great necessity to create the new strategy.

LITERATURE SURVEY

Wairojjana et.al.[18] various mathematical programming applications can be formulated as a variational inequality model, such as minimax issues, penalization methods and fixed-point issues. Most of the methods used to solve such problems include iterative algorithms, so we are implementing a new extra gradient-like method in this paper to solve the problems of variational inequalities involving pseudo monotone operators in real Hilbert space. The approach has a strong advantage because of a variable stepsize formula that is updated for each iteration based on the previous iterations. The main benefit of the technique is that it operates without the Lipschitz constant's prior knowledge. Under mild conditions, strong convergence of the system is confirmed. Several numerical tests are recorded to illustrate the method's numerical behavior.

Zhao et.al.[19] discuss the monotonous variational inequalities and fixed point issues in Hilbert spaces in this paper. Two modified extra gradient algorithms are presented to find a common element of the set of fixed points of a pseudocontractive operator and the set of solutions to the variational inequality problem. The proposed algorithms show weak and strong convergence.

zuchukwu et.al.[20] reported that for finite groups of variational inequalities and the split equality fixed-point question, the main objective is to implement an iterative algorithm for approximating a common solution to a split equality problem. By using the iterative algorithm, they state and demonstrate a clear convergence theorem for the approximation of an element at the intersection of the set of split-equality problem solutions for finite families of variational inequalities and the set of fixed-point problem split equality solutions for countable families of demi contractive type-one multivalued mappings. Finally, they apply the outcome to issues specific to the analysis.

Alakoya et.al.[21] reported the setting of Hilbert space with a typical monotone and Lipschitz continuous variational inequality and fixed-point issues can be defined on a level set of a convex function. An updated extra gradient inertial viscosity subgradient algorithm with self-adaptive step size is proposed in which the two projections are rendered on some half-spaces. In addition, under some mild conditions, they obtain a good convergence result for approximating a common solution of the variational inequality and fixed point of quasi-non-expansive mappings. The key advantages of the approach are the self-adaptive step size that eliminates the need to know apriori of the related monotone operator's Lipschitz constant, the two projections made on certain half-spaces, the high convergence and the inertial technique used to speed up the algorithm's rate of convergence.

Ghoussoub et.al.[22] examine the profile of one-step martingale plans on R^d x R^d that maximize the expected value of the modulus of their increment among all martingales having and as marginals, given two prospect measures v and μ in convex order on R^d. Whereas here is an immense contract of results for the real line (i.e., when d = 1), much less is known in the better-off and more fragile upper dimensional case that they tackle in this paper. They demonstrate that, assuming the initial measure is continuous with respect to the Lebesgue measure, several structural conclusions may always be obtained anytime a natural dual optimization problem is attained. One such property is that μ-almost every x in R^d is transported by the optimal martingale plan into a probability measure π_x concentrated on the extreme points of the closed convex hull of its support. This will be established in full generality in the 2-dimensional case and for any d ≥ 3 as long as the marginals are in "subharmonic order". Sometimes, π_x is supported on the vertices of a k(x)-dimensional polytope, such as when the target measure is discrete. Many proofs rely on a remarkable decomposition of "martingale supporting" Borel subsets of $R^d \times R^d$ into a collection of mutually disjoint components using a "convex paving" of the source space. If the martingale is optimal, then each component in the decomposition supports a restricted optimal martingale transport for which the dual problem is attained.

Muu et.al.[23] presented the relationship between the fixed points of the Moreau proximal mapping and the equilibrium problem solutions that satisfy some kinds of monotonicity and Lipschitz-type condition. This relationship allows the equilibrium problem solved by fixed point theory since in the fixed-point theory has iterative methods for computing a fixed point, which has been successfully applied to contractive, generalized contractive, and no expansive mappings. However, the bi function involved in this method is quasi-convex for its second variable.

Saboksayr et.al.[24] assume that signals from each class are smooth for its corresponding graph while remaining non-smooth concerning the graphs from other classes. The learned representations' discriminative features are retrieved using the graph Fourier transform (GFT) and applied to subsequent learning challenges. Second, we broaden our research to include real-time topological inference and more dynamic situations. To do this, we use time-varying convex optimization and recent GSP advancements. We create a proximal gradient (PG) method that is flexible enough to be applied in scenarios where data are collected instantly. However, the time-varying graphs inference from streaming signals abrupt the connectivity.

Wang et.al.[25] created a novel method to assess the reliability of vibration data. To ensure the efficiency of the subsequent computation, the raw vibration data are first converted directly into time-frequency images using a tensor format and saved. Then, an off-the-shelf pre-trained Inception model is used to represent their high-order data structure and amplitude-wise dependence. To assess the similarity between the real data and created data in a high-dimensional feature space, the NSD metric is constructed and applied. However, the data quality should be enhanced to improve the efficiency.

Hence many reports do not involve computing the projection,[18] intersections arise the split equality convex minimization crisis, [19] require the prior knowledge of the operator norms,[20,21] real world applications it is severely hindered by its computational burden insists the non-regularization of transport problem,[22] bi function is quasi convex,[23] the streaming signals abrupt the connectivity[24] and data quality should be enhanced.[25] Though there is a great essential to develop a novel strategy to tackle those issues and attains the strong convergence, greater more consistency and accuracy in the fixed point theory applications.

FIXED POINT THEORY ACCOMPLISHMENTS

In the last few decades, the fixed-point theory has seen many applications. Its implications are quite interesting and informative for the theory of optimization, game theory, conflict situations, and quality mathematical modeling and its management. Most of the prior literature portrayed these considerations, and also under some mild assumptions in operators that obtained weak convergence, poor consistency, and accuracy, as well which does not involve computing the projection onto the intersections, thus significantly attaining the variation inequality problem (VIP) and difficult to find the fixed point of demi contractive mapping in a real Hilbert space. While some of the studies solve VIP and tackle the major obstacles in fixed point, the split equality convex minimization crisis arises because existing techniques require prior knowledge of the operator norms. In addition, despite its conceptual appeal and its first practical success in various (statistical) applications, the routine use of optimal transport-based data analysis is still lacking. For many real-world applications, it is severely hindered by its computational burden that insists the non-regularization of transport problems. From the contemplation mentioned above, it is clear that to overwhelm the most significant hindrances in the embryonic field of applied mathematics with fixed point theory are variation inequality problem, difficulty in finding the fixed point of demi contractive mapping, split equality convex minimization crisis, non-regularization of transport problem. To solve this, the present research is most essential to develop a novel strategy.

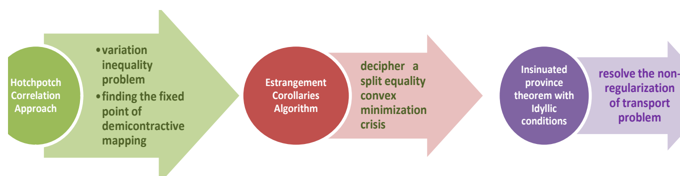

Figure 1: Flow of concept Block diagram

As fixed-point theory plays a significant part in mathematics, which includes economic theories, mathematical models, and optimization, mathematics is the art of giving the same name to several entities. Here, existing strategies still have not solved major interferences such as variation inequality problem, difficulty finding the fixed point of demi-contractive mapping, split equality convex minimization crisis, and non-regularization of transport problem. To deal with these issues, the proposed work introduces competent novel strategies, initially that include a new Hotchpotch Correlation Approach for solving the variation inequality problem and finding the fixed point of demi contractive mapping. The aspect of this approach is that in any iteration, a single projection is needed onto the feasible set, and the phase size for the next iterate is calculated so that a prior approximation of the underlying operator's Lipschitz constant is not required. In some optimal conditions on the influence variables, the work recognizes and reveal a strong convergence theorem for approximating effective solution of variational inequality and fixed points problem. After solving the VIP and fixed points problem, more essential to decipher a split equality convex minimization crisis, the research incorporates Estrangement Corollaries Algorithm, which introduces a generalized step size such that the algorithm does not require a prior knowledge of the operator along with the algorithm that generated a sequence. The algorithm also integrates some application results on inverse problems and numerical examples to show the consistency and accuracy of the proposed work. To resolve the non-regularization of transport problem, the research includes an insinuated province theorem with essential and adequate idyllic conditions. The theorem derives the restricted distributions amongst probability distributions endorsed on a fixed metric space and focuses on specific consistency, including its bootstrap for empirical regularized optimal transport distances. In particular, the work reveals that a Gaussian law is asymptotically followed in the empirical regularized transport plan system. The theory involves regularizing considerable entropy, thus a limitation law for the extensively permissible Sinkhorn divergence. Consequently, the suggested strategies provide a way to establish the existence of a solution to a set of equations via resolving the variation inequality problem, difficulty in finding the fixed point of demi contractive mapping split equality convex minimization crisis, and non-regularization of transport problem. That demonstrates the efficacy of the proposed work in a cost-effective and not as complex manner.

Hotchpotch Correlation Approach:

In this section, the work described the solution of the variation inequality problem and found the fixed point of demi-contractive mapping.

Let D be a nonempty closed and convex subset of a real Hilbert space I. and let B: I→I be a pseudo monotone and L- L-Lipschitz continuous operator and U: I →I be a α demi contractive mapping with constant α € [0, 1] and demi closed at zero. Suppose $sol := WJ(D,B) \bigcap G(U) \neq 0$, allow C: I→I be a k-Lipschitzian and β-strongly monotone mapping with l > 0 and β > 0 and g: I → I be a η-Lipschitz mapping with η > 0. Let $0 < \delta_1 < \frac{2\beta}{l^2}$ and, $0 < \zeta\eta < \sigma$, and where $\sigma = \frac{1}{2}\delta\left(2\beta - \delta l^2\right)$. Let $\{\gamma_l\}$ and $\{w_l\}$ are sequences in (0, 1) and $\{y_l\}$ are generated based on the following algorithm:

Algorithm 3.1

Step 0: Select the preliminary information y_1 € I and parameters ϕ, α € (0, 1), λ € (0, 2). Locate l=1.

Step 1: Estimate

yk = PC(xk − λk Axk),

where λk = γ lk , and lk is the smallest nonnegative integer satisfying

λk ||A(xk) − A(yk)|| ≤ θ||xk − yk ||. (3.2)

Step 2: Compute

d (xk , yk) = xk − yk − λk (Axk − Ayk), (3.3)

wk = xk − σ δkd(xk , yk), (3.4) where

δk = xk − yk , d(xk , yk) ||d(xk , yk)||2 ,

if d(xk , yk) = 0, 0,

if d(xk , yk) = 0. (3.5)

Step 3: Compute

xk+1 = αk ξ f (xk) + (I − αkμB)(vkT wk + (1 − vk)wk). (3.6)

Set k := k + 1 and go to Step 1.

To establish the convergence of Algorithm 3.1, we make the following assumption:

(C1) limk→∞ αk = 0 and ∞ k=0 αk = ∞;

(C2) lim infk→∞ λk > 0;

(C3) lim infk→∞(vk − β)vk > 0.

Remark 3.2 Observe that if xk = yk and xk − T xk = 0, we are at a common solution of the variational inequality (1.1) and fixed point of the demi contractive mapping T. In our convergence analysis, we will implicitly assume that this does not occur after finitely much iteration so that our Algorithm 3.1 generates infinite sequences. We will see in the following result that Algorithm 3.1 is well-defined. To do this, it suffices to show that the Armijo line searching rule defined by (3.2) is well defined and δk = 0.

Lemma 3.3 There exists a nonnegative integer lk satisfying (3.2).

In addition δk ≥ 1 − θ (1 + θ)2 .

Proof Let rλk (xk) = xk − PC(xk − λk Axk) and suppose rγ k0 (xk) = 0 for some k0 ≥ 1. Take lk = k0, which satisfies (3.2). Suppose rγ k1 (xk) = 0 for some k1 ≥ 1 and assume the contrary, that is,

γ l ||Axk − A(PC(xk − γ l Axk))|| > θ||rγ l(xk)||.

Then it follows from Lemma 2.9 and the fact that γ ∈ (0, 1) that

||Axk − A(PC(xk − γ l Axk))|| > θ γ l ||rγ l(xk)||

≥ θ γ l min{1, γ l }||r1(xk)||

= θ||r1(xk)||.

Since PC is continuous, we have that

PC(xk − γ l Axk) → PC(xk), l → ∞.

We now consider two cases: when xk ∈ C and xk ∈/ C.

(i) If xk ∈ C, then xk = PC xk . Now since rγ k1 (xk) = 0 and γ k1 ≤ 1, it follows from Lemma 2.9 that

0 < ||rγ k1 (xk)|| ≤ max{1, γ k1 }||r1(xk)||

= ||r1(xk)||.

Letting l → ∞ in (3.8), we have that

0 = ||Axk − Axk || ≥ θ||r1(xk)|| > 0.

This is a contradiction, and so (3.2) is valid.

(ii) xk ∈/ C, then

γ l ||Axk − Ayk || → 0, l → ∞,

While,

lim l→∞ θ||rγ l(xk)|| = lim l→∞ θ||xk − PC(xk − γ l Axk)|| = θ||xk − PC xk || > 0.

This is a contradiction. Therefore, the Armijo line searching rule in (3.2) is well defined. On the other hand, since A is Lipschitz continuous, then we have from (3.2) and (3.3):

xk − yk , d(xk , yk)=xk − yk , xk − yk − λk (Axk − Ayk)

= ||xk − yk ||2 − λk xk − yk , Axk − Ayk

≥ ||xk − yk ||2 − λk ||xk − yk ||||Axk − Ayk ||

≥ ||xk − yk ||2 − θ||xk − yk ||2

= (1 − θ)||xk − yk ||2. (3.10)

Also,

||d(xk , yk)|| = ||xk − yk − λk (Axk − Ayk)||

≤ ||xk − yk || + λk ||Axk − Ayk || ≤ (1 + θ)||xk − yk ||. (3.11)

Therefore from (3.5), (3.10) and (3.11), we get

δk = xk − yk , d(xk , yk) ||d(xk , yk)||2 ≥ (1 − θ) (1 + θ)2 .

Estrangement Corollaries Algorithm

This section presents a modified Halpern algorithm for solving (1.14) where T1 and T2 are Bregman quasi-nonexpansive mappings. Theorem 3.1 Let E1, E2 and E3 be p-uniformly convex real Banach spaces, which are also uniformly smooth. Let C and Q be nonempty closed, convex subsets of E1 and E2, respectively, A: E1 → E3 and B: E2 → E3 be bounded linear operators. Let f : E1 → R ∪ {+∞} and g: E2 → R ∪ {+∞} be proper, convex and lower semicontinuous functions, T1: E1 → E1 and T2: E2 → E2 be Bregman quasi-nonexpansive mappings such that = ∅. For fixed u ∈ E1 and v ∈ E2, choose an initial guess (x1, y1) ∈ E1 × E2 and let {αn}⊂[0, 1]. Assume that the nth iterate (xn, yn) ⊂ E1 × E2 has been constructed; then we compute the (n + 1)th iterate (xn+1, yn+1) via the iteration:

un = proxγn f J E∗ 1 q J E1 p (xn) − γn A∗ J E3 p (Axn − Byn) , xn+1 = J E∗ 1 q αn J E1 p (u) + (1 − αn) βn J E1 p (un) + (1 − βn)J E1 p (T1un) , vn = proxγn g J E∗ 2 q J E2 p (yn) + γn B∗ J E3 p (Axn − Byn) , yn+1 = J E∗ 2 q αn J E2 p (v) + (1 − αn) δn J E2 p (vn) + (1 − δn)J E2 p (T2vn) , (3.1)

For n ≥ 1, {βn}, {δn} ⊂ (0, 1), where A∗ is the adjoint operator of A. Further, we choose the stepsize γn such that if n ∈ := {n: Axn − Byn = 0}, then γ q−1 n ∈ 0, qAxn − Bynp Cq A∗ J E3 p (Axn − Byn)q + Dq B∗ J E3 p (Axn − Byn)q, (3.2

Cq and Dq are constants of smoothness of E1 and E2, respectively. Otherwise, γn = γ (γ being any nonnegative value). Then {xn} and {yn} are bounded. Proof Let (x, y) ∈, using Lemma 2.1, (2.3) and the Bregman firmly nonexpansive of prox operators, we have

p(x, un) = p x, proxγn f J E∗ 1 q J E1 p (xn) − γn A∗ J E3 p (Axn − Byn)

≤ p x, J E∗ 1 q J E1 p (xn) − γn A∗ J E3 p (Axn − Byn) = xp p − x, J E1 p xn + γn x, A∗ J E3 p (Axn − Byn)

= p(x, xn) − γn Axn − Ax, J E3 p (Axn − Byn) + Cq q γ q n A∗ J E3 p (Axn − Byn)q .

Following similar to the argument as in (3.3), we have

p (y, vn) ≤ p(y, yn) + γn Byn − By, J E3 p (Axn − Byn) + Dq q γ q n B∗ J E3 p (Axn − Byn)q.

Adding (3.3) and (3.4) and noting that Ax = By, we obtain

p(x, un) + p(y, un) ≤ p(x, xn) + p(y, yn) − γn Axn − Byn, J E3 p (Axn − Byn)

Thus, the last inequality implies that {xn} and {yn} are bounded. Consequently, {un}, {vn}, {T1un} and {T2vn} are bounded. Theorem 3.2 Let E1, E2 and E3 be p-uniformly convex real Banach spaces, which are also uniformly smooth. Let C and Q be nonempty closed, convex subsets of E1 and E2, respectively, A: E1 → E3 and B: E2 → E3 be bounded linear operators. Let f : E1 → R ∪ {+∞} and g: E2 → R∪{+∞} be proper, convex and lower semicontinuous functions, T1: E1 → E1 and T2: E2 → E2 be Bregman quasi-nonexpansive mappings such that F(Ti) = Fˆ(Ti), i = 1, 2

and = ∅. For fixed u ∈ E1 and v ∈ E2, choose an initial guess (x1, y1) ∈ E1 × E2 and let {αn}⊂[0, 1]. Suppose ({xn},{yn}) is generated by algorithm (3.1) and the following conditions are satisfied: (i) limn→∞ αn = 0, and ∞ n=0 αn = ∞, (ii) 0 < a ≤ lim infn→∞ βn ≤ lim supn→∞ βn < 1, (iii) 0 < b ≤ lim infn→∞ δn ≤ lim supn→∞ δn < 1.

Case 1 Suppose ∃ n0 ∈ N such that {n} is monotonically non-increasing for all n ≥ n0. Since n is bounded, it implies that {n} converges and n+1 − n → 0, as n → ∞.

Set Kn = Cq A∗ J E3 p (Axn − Byn)q + Dq B∗ J E3 p (Axn − Byn)q ,

it follows from (3.10) that

γn(1 − αn) Axn − Bynp − γ q−1 n q Kn ≤ (1 − αn)n − n+1 + αnτn → 0, (3.11) as n → ∞. By the choice of the stepsize (3.2), there exists a very small > 0 such that

0 < γ q−1 n ≤ q||Axn − Byn||p Kn − ,

This means that

γ q−1 n Kn ≤ q||Axn − Byn||p − Kn,

And hence

Kn q ≤ ||Axn − Byn||p − γ q−1 n q Kn → 0, as n → ∞.

Hence

lim n→∞ Kn = lim n→∞ Cq A∗ J E3 p (Axn − Byn)q + Dq B∗ J E3 p (Axn − Byn)q = 0.

Insinuated Province Theorem

For two probability distributions r, s ∈ ∆N , parameters λ > 0, p ≥ 1 and proper regularizer f an estimator for πp,λ,f (r, s) in (2.2) is given by its empirical counterpart πp,λ,f (ˆrn, s) with ˆrn the empirical distribution of the i.i.d. sample X1, . . . , Xn in (1.6). The next theorem states a Gaussian limit distribution for the empirical ROT plan. Since the sensitivity result in Theorem 2.3 holds regardless of r = s or r 6= s and as the ROT plan is always dense, we do not derive any substantial difference regarding statistical limit behavior in either of these cases.

Theorem 3.1. Let r, s ∈ ∆N be two probability distributions on the finite metric space (X , d) and let rˆn be the empirical version given in (1.6) derived by X1, . . . , Xn i.i.d. ∼ r. Then, as the sample size n grows to infinity, it holds that

√ n {πp,λ,f (ˆrn, s) − πp,λ,f (r, s)} D−→ NN2 (0, Σp,λ,f (r|s))

with covariance matrix

Σp,λ,f (r|s) = ∇r φp,λ,f (r, s?) Σ(r) [∇r φp,λ,f (r, s?)]T , (3.1)

where Σ(r) is defined in (1.10) and ∇r φp,λ,f (r, s?) are the partial derivatives of φp,λ,f with respect to r as given in Theorem 2.3.

The proof is based on the multivariate delta method and straightforward given Theorem 2.3, hence postponed to the supplement. Further, we prove limit distributions for the empirical counterpart of the ROT distance (1.5). Here, s (which might be equal to r) plays the role of a fixed reference probability distribution to be compared empirically with the probability distribution r. The proof is again an application of the delta method in conjunction with the limit law from Theorem 3.1. We again do not derive any substantially different distributional limit behavior between the cases r = s and r 6= s. This is in notable contrast to the non-regularized OT

Theorem 3.2. Under the assumptions of Theorem 3.1, as n → ∞ it holds that

√ n {Wp,λ,f (ˆrn, s) − Wp,λ,f (r, s)} D−→ N1 0, σ2 p,λ,f (r|s)

with variance

σ 2 p,λ,f (r|s) = γ T Σp,λ,f (r|s) γ , (3.2)

where γ is the gradient of the function π 7→ hcp, πi 1 p evaluated at the regularized transport plan πp,λ,f (r, s), and Σp,λ,f (r|s) is the covariance matrix from Theorem 3.1. Standardizing by the square root of the empirical variance σ 2 p,λ,f (ˆrn|s) results in a standard normal limit distribution. As a corollary, we immediately obtain limit distributions for the empirical entropy ROT plan and the Sinkhorn divergence.

Corollary 3.3 (Sinkhorn Transport and Sinkhorn Divergence). Consider the negative Boltzmann-Shannon entropy f in (1.4). Then, the statements in Theorem 3.1 and Theorem 3.2 remain valid. Note that the gradient inherent in the corresponding covariance matrix (3.1) is given by Example 2.6.

Remark 3.4 (Entropy ROT Type Functionals). From Theorem 3.1, we easily derive asymptotic distributions for any sufficiently smooth function of the ROT plan. Exemplarily, we consider the objective function in (1.3) denoted as d(πp,λ,f (r, s)). A straightforward calculation shows that

∇d(πp,λ,f (r, s)) = cp + λ log(πp,λ,f (r, s)) = (αp,λ,f , βp,λ,f ?)A?

The second equality follows by primal-dual optimality relation between πp,λ,f and its optimal dual solutions (αp,λ,f , βp,λ,f ?) [26] with lower subscript star as we delete the last constraint in (1.3)

(Remark 2.1): In conjunction with Example 2.6 and Theorem 3.1 we conclude

√ n {d(πp,λ,f (ˆrn, s)) − d(πp,λ,f (r, s))} D−→ hG, αp,λ,f i , (3.3)

where G ∼ NN (0, Σ(r)) [27]. Notably, if r = s the limit law in (3.3) is non-degenerate. This is not true anymore for the Sinkhorn loss [28] defined by

Sλ(r, s) := d(πp,λ,f (r, s)) − 1 2 (d(πp,λ,f (r, r)) − d(πp,λ,f (s, s))) ,

As then ∇Sλ(r, r) = 0 .[27] However, a second-order expansion based on a perturbation analysis for the dual solutions provides a non-degenerate asymptotic limit of nSλ(ˆrn, r). This can be represented as a weighted sum of independent χ 2 1 random variables. Exact computation is tedious but follows the lines of G. Luise et.al,[29] who also provide a perturbation analysis for the dual solutions. The weights of this sum are then given by the eigenvalues of the Hessian ∇2Sλ(r, r). From this, it can be shown that the m out of n bootstrap is consistent when m = o(n) [30], which is an alternative to the bootstrap suggested by J. Bigot et.al. report.[27]

Result And Discussion

This section ensures the validity of our proposed study by reviewing the findings and contrasting the proposed studies to previous methodologies. This section aims to show the outcomes of enhanced strong convergence, attaining greater consistency and accuracy.

System Specification

The proposed system has been implemented in MATLAB

Platform	:	MATLAB
OS	:	Windows 7
Processor	:	Intel core i5
RAM	:	8GB RAM

Performance Evaluation

In this section, we present three numerical examples which demonstrate the performance of our Algorithm 3.1. Let U: I → I be defined by

$$U_y = \begin{cases} -\frac{9}{2}, & if\ y \le 0 \\ -2y, & if\ y > 0 \end{cases}$$

It is easy to see that U is demicontractive mapping with $\alpha = \frac{77}{121}$, $G(U) = \{0\}$, *we let* $g = J$, $C = \frac{1}{2}J$, *then* $\eta = 1$ *and* $\beta = 1 = l$, *Hence* $0 < \delta < \frac{2\beta}{l^2} = 2.$ *Let us choose* $\delta = 1$ *so that* $\sigma = \frac{1}{2}\delta(2\beta - \delta l^2) = 1$. *As* $0 < \xi_\eta < \sigma$, *we have* $\xi \varepsilon (0,2)$ *without loss of generality we choose* $\xi = 1$

In each example, we fix the stopping criterion as $\| x_{k+1} - x_k \| = \varepsilon < 10^{-5}$, $\sigma = 0.7$, $\gamma = 0.54$, $\lambda_k = = 0.15$, *and let* $\alpha_k = \frac{1}{k+1}$, $w_k = \frac{2k+3}{4k+12}$. *The projection onto the feasible set D is carried out using MATLAB solve and projection onto an hyperlane* $Q = \{ y \varepsilon I : < a, x > = 0\}$ *is defined by*

$$P_Q(x) = x - \frac{< a, x >}{\| a \|^2} a.$$

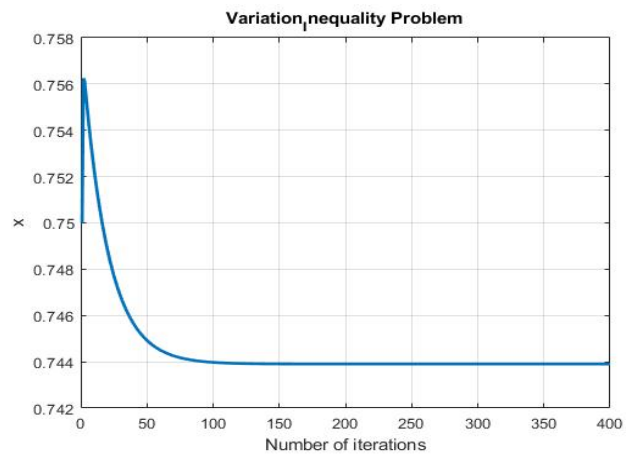

Figure 2: variation inequality for number of iterations with x fixed point

Figure 2 shows the variation inequality in a number of iterations with x fixed point. From the graph, when the iteration is 0, the value of x increases from 0.75 to 0.756 and then decreases rapidly between 0th and 75th iterations. After the 75th iteration, the value of x remains constant. Thus, the x fixed point is identified using the proposed Hotchpotch Correlation Approach, which mitigates the variation inequality problem and improves the theory's performance.

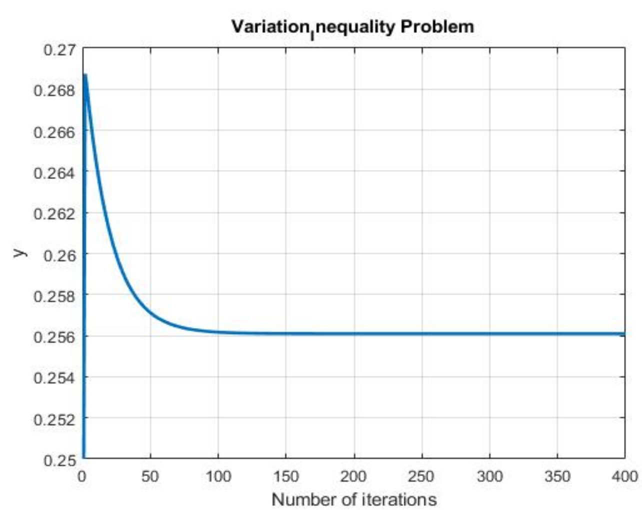

Figure 3: variation inequality in number of iterations with y fixed point

Figure 3 shows the variation inequality in a number of iterations with y fixed point before Hotchpotch Correlation Approach is applied. From the graph, when there is no iteration, the value y ranges from 0.25 to 0.269. when the iteration increases, the value of y decreases rapidly till 100th iteration. From 100th iteration the value y remains constant. Thus the y fixed point is identified by this proposed Hotchpotch Correlation Approach, which mitigates the variation inequality problem and improves the theory's performance.

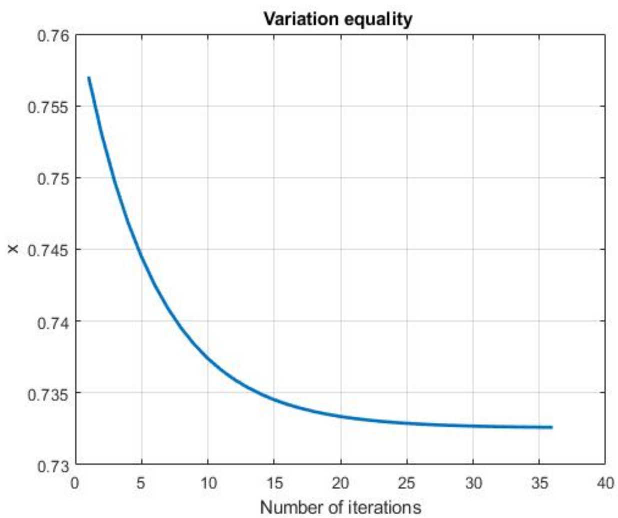

Figure 4: variation equality in number of iterations with x fixed point

Figure 4 shows the variation equality in number of iterations with x fixed point after the Hotchpotch Correlation Approach applied. The graph shows that the value x decreases with an increase in number of iterations. When x = 0.733 the number of iterations is 35. Thus, the number of iterations reaches an x fixed point based on the nature of the equation in the proposed Hotchpotch Correlation Approach and the convergence properties of the iteration. This improves the performance of the fixed-point theory.

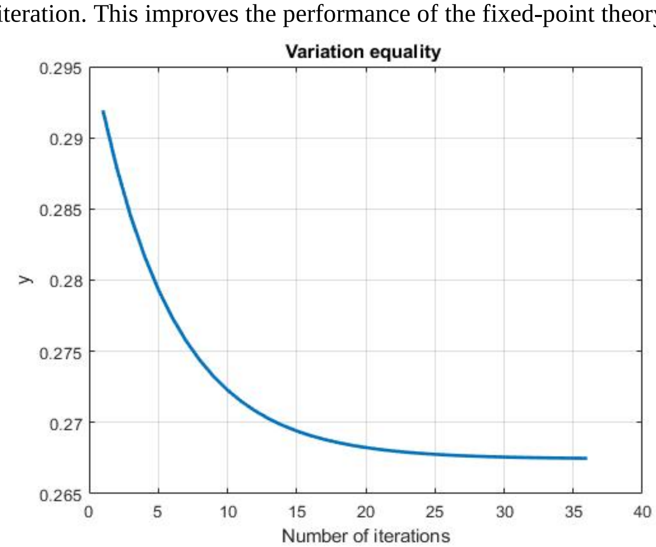

Figure 5: variation equality in number of iterations with y fixed point

Figure 5 shows the variation equality in number of iterations with y fixed after the Hotchpotch Correlation Approach applied. From the graph, the value y decreases with an increase in the number of iterations. When y = 0.268 the number of iterations is 35, thus the number of iterations reaches a y fixed point based on the nature of the equation in the proposed Hotchpotch Correlation Approach and the convergence properties of the iteration. This improves the performance of the fixed point theory.

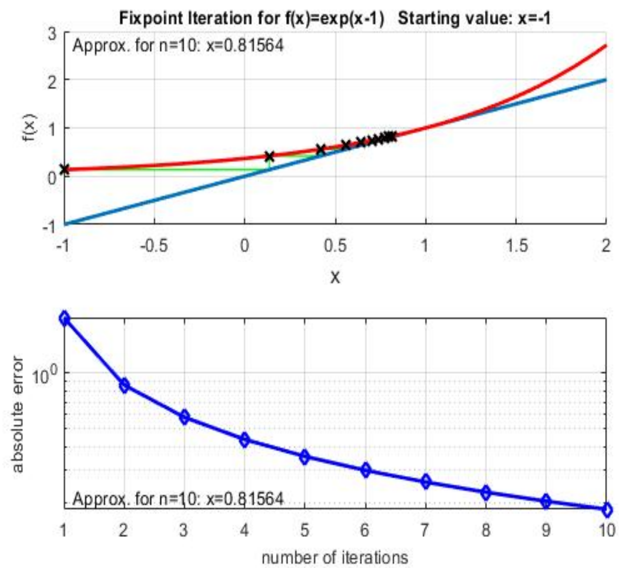

Figure 6: Fixed Point Iteration for function f(x)=exp(x-1)

Figure 6 shows the Fixed-Point Iteration for function f(x)=exp(x-1), considering the starting value of x is -1, approximately the value of n = 10, x = 0.81564. From the graph, the absolute error decreases with an increase in iterations. So, this represents that the initial guess is negative in the exponential equation, reducing the absolute error in this method and improving the system's efficiency.

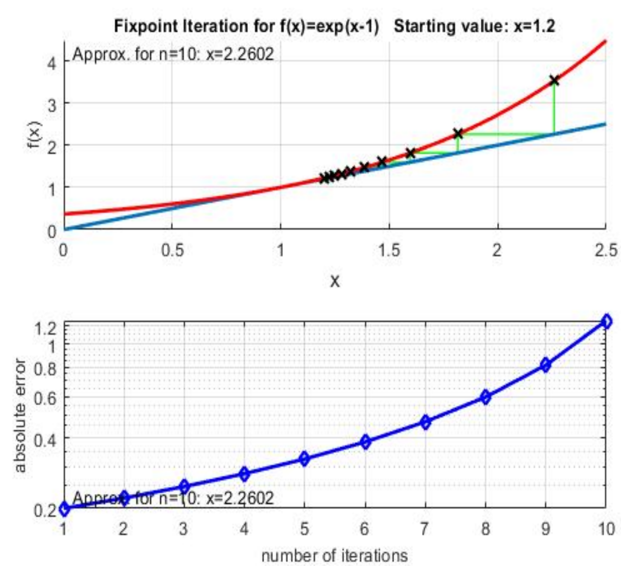

Figure 7: Fixed Point Iteration for function f(x)=exp(x-1)

Figure 7 shows the Fixed-Point Iteration for function f(x)=exp(x-1), considering the starting value of x is 1.2, approximately the value of n = 10, x = 2.2602. From the graph, the absolute error increases with an increase in iterations. Thus, the fixed point iteration in this function is with positive initial guess, increasing the absolute error, which this proposed approach reduces.

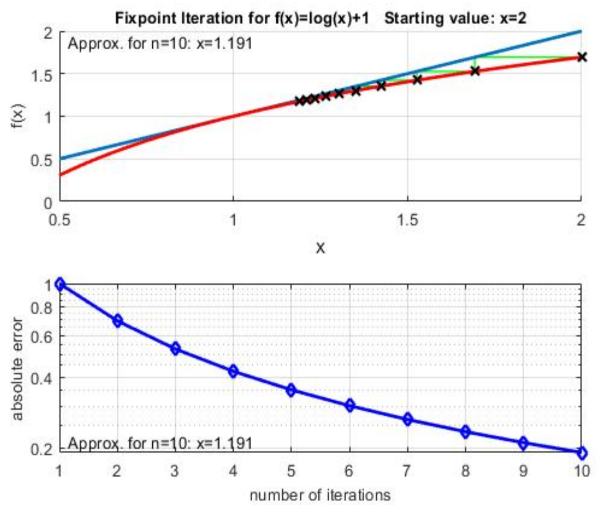

Figure 8: Fixed Point Iteration for function f(x) = log (x) + 1

Figure 8 shows the Fixed-Point Iteration for function f(x) = log(x) + 1, considering the starting value of x is 2, approximately the value of n = 10, x = 1.191. From the graph, the absolute error decreases with an increase in iterations. So this represents that the initial guess in this equation reduces the absolute error in this method and improves the system's efficiency by finding the fixed point for convergence.

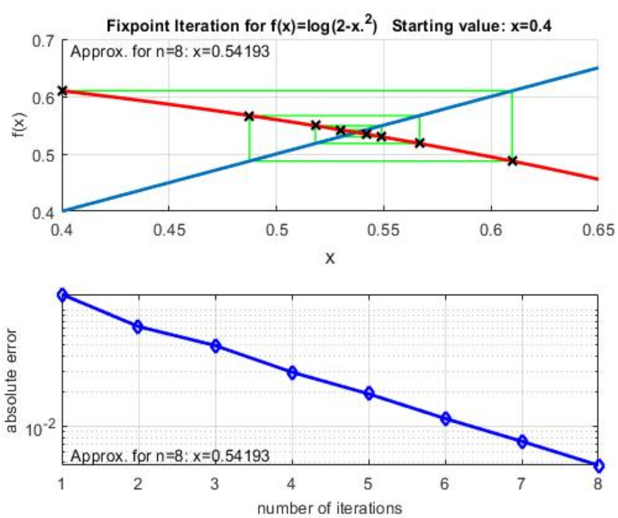

Figure 9: Fixed Point Iteration for function f(x) = log (2 – x2)

Figure 9 shows the Fixed-Point Iteration for function f(x) = log $(2 - x^2)$ considering the starting value of x is 0.4, approximately the value of n = 8, x = 0.54193. From the graph the absolute error decreases with an increase in iterations. This indicates that the first guess in the equation decreased the method's absolute error and increased the system's effectiveness by locating the fixed point for convergence.

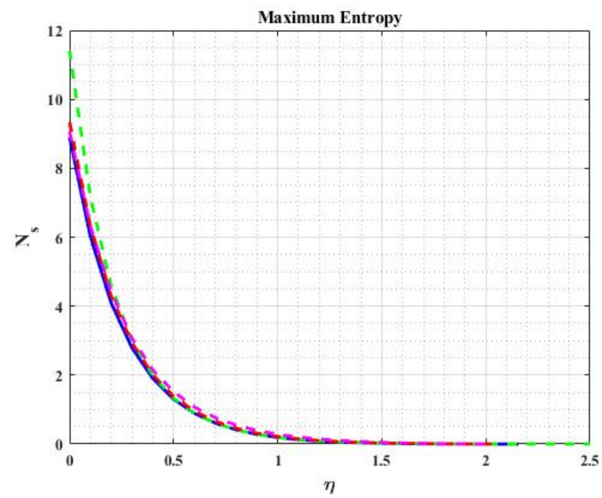

Figure 10: Maximum Entropy

Figure 10 shows the maximum entropy for different values of η. When η = 2.5 the value of Ns = 11.5, similarly when η= 2.20, 2 the value of Ns is 9, 9.5, respectively. The empirical transport regularization finds the solution that balances data fidelity with maximum entropy. This helps in obtaining more stable and accurate solutions, even in the presence of noisy data.

Conclusion

In the last few decades, the fixed-point theory has seen many applications. Its implications are quite interesting and informative for the theory of optimization, game theory, conflict situations, and quality mathematical modeling and its management. Hence, the work efficiently tackles the major significant issues such as the variation inequality problem (VIP) and difficulty finding the fixed point of demi contractive mapping in a real Hilbert space. As a result, the proposed strategies provide a way to prove the existence of a solution to a set of equations by resolving the variation inequality problem, the difficulty of finding the fixed point of demi contractive mapping, the split equality convex minimization crisis, and the non-regularization of the transport problem. The experimental results show that the novel approach proves the intended work's efficacy in a cost-effective and less complicated method.

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